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ONE DIMENSIONAL FLOW OF A GAS PARTICLE SYSTEM

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TABLE OF CONTENTS

LIST OF FIGURES

SUMMARY

A family of solutions to the equations governing the one dimensional flow of a gas-particle system are presented. The basic assumptions upon which the theory is based are that the particles do not interact (distance between particles large compared to their size) and that the particles are spheres and are always in Stokes flow regime (small particle lags). It is shown that this family of solutions corresponds to flow in a nozzle in which the axial velocity gradient is a constant. As this condition is approximately realized in the throat region of most nozzles, it is believed that the theory has general validity near the nozzle throat. The theory shows that:

1) If the particles do not undergo a phase change, the solutions for the one dimensional flow of a pure gas are identical to the above family of solutions if a suitably modified Mach number and specific heat ratio are chosen.

2) For a given particle size, the particle velocity lag (the ratio of the particle velocity to the gas velocity) is determined by the axial gas velocity gradient.

3) The particle thermal lag (the ratio of the change in particle temperature to the change in gas temperature) is related to the particle velocity lag. In systems of engineering interest, the particle thermal lag is always greater than the particle velocity lag.

4) If the particles solidify, the gas undergoes an isothermal expansion only if the particles are in velocity and thermal equilibrium with the gas.

The design of a nozzle for use with a gas-particle exhaust system is discussed. It is concluded that:

1) The proper design of a nozzle for a gas-particle system should begin at the combustion chamber. The convergent section has a measurable effect on the performance of the system.

2) The throat region is the critical region in the design of a gas-particle nozzle as the gas velocity gradient is highest in this region. The throat region essentially determines the particle lags for the nozzle and hence its performance.

3) In order to obtain low particle lags and hence high performance, the nozzle as a whole must be lengthened, especially the throat region.

4) The wall radius of curvature at the throat should be continuous. A performance degradation results if the wall radius of curvature is smaller downstream than upstream as is common in the present day design of bell shaped gas nozzles.

5) The area ratio at which particle solidification occurs increases as the particle lag increases. This results in a performance degradation through inefficient use of the particles heat of fusion.

6) Nozzle length is an important parameter which must be considered in subscale nozzle tests. On a one dimensional basis, two nozzles are similar if they are the same length and have the same area ratio variation with length.

INTRODUCTION

The use of propellants with metal additives requires the design of a nozzle which is optimized for the gas-particle exhaust of such fuels. Experience has shown that rather large performance degradations can result through the use of an optimum gas nozzle with metallized propellants. The design of an optimum nozzle for use with a gas particle system will be approached from a general viewpoint, without reference to experimental data. A family of solutions for the one dimensional flow of a gas-particle system will be presented which are believed to have general validity in the region of the nozzle throat. The design of an optimum nozzle for use with a gas-particle system will be discussed as will the differences between an optimum gas-particle and pure gas nozzle.

GENERAL ONE DIMENSIONAL GAS-PARTICLE FLOWS

Consider the steady one dimensional flow of a gas-particle system. The following assumptions will be made:

1) There are no mass or energy losses from the system.

2) The particles do not interact and may be considered identical spheres of constant size.

3) The thermal (Brownian) motion of the particles does not contribute to the pressure of the system.

4) The particles have a large thermal conductivity so that their internal temperature is uniform.

5) Energy exchange occurs between the particles and the gas only by convection.

6) The gas may be considered a perfect gas of constant composition.

7) The gas is inviscid except for the drag it exerts on the particles.

8) The physical properties of the gas and particles are constant.

Within the above assumptions, the equations governing the steady one dimensional flow of a gas-particle system are the following:*

Continuity equations

$$\rho_g u_g A = \dot{w}_g \tag{1}$$

$$\rho_p u_p A = \dot{w}_p \tag{2}$$

* See nomenclature list for definition of all symbols used in the text.

Momentum equation

$$\rho_g u_g \frac{du_g}{dx} + \rho_p u_p \frac{du_p}{dx} + \frac{dP_g}{dx} = 0 \tag{3}$$

Energy equation

$$\dot{w}_g \left[C_{pg}(T_g - T_{g0}) + \frac{1}{2} u_g^2 \right] + \dot{w}_p \left[(h_p - h_{p0}) + \frac{1}{2} u_p^2 \right] = 0 \tag{4}$$

Equation of state of the gas

$$P_g = \rho_g R T_g \tag{5}$$

Equation of particle motion

$$u_p \frac{du_p}{dx} = \frac{3}{8} \frac{C_D \rho_g}{m_p r_p} (u_g - u_p)|u_g - u_p| \tag{6}$$

Equation of particle thermal energy loss

$$u_p \frac{dh_p}{dx} = -\frac{3h}{m_p r_p} (T_p - T_g) \tag{7}$$

In addition to the above seven equations, the relation between the particle heat content (h_p) and the particle temperature (T_p) must be specified. This will be done later as various special cases are considered.

Eliminating ρ_g, ρ_p, T_g and P_g by means of equations (1), (2), (4) and (5), we obtain the following set of first order, non-linear ordinary differential equations for U_g, U_p and h_p:

$$\frac{RT_g}{u_g}(M^2 - 1)\frac{du_g}{dx} + \frac{\dot{w}_p}{\dot{w}_g}\left[\left(u_g - \frac{R}{C_{pg}} u_p\right)\frac{du_p}{dx} - \frac{R}{C_{pg}}\frac{dh_p}{dx} \right] - \frac{RT_g}{A}\frac{dA}{dx} = 0 \tag{8}$$

$$u_p \frac{du_p}{dx} = \frac{3}{8} \frac{\dot{w}_g C_D}{u_g A m_p r_p} (u_g - u_p)|u_g - u_p| \tag{9}$$

$$u_p \frac{dh_p}{dx} = -\frac{3k}{m_p r_p}\left(T_p - T_g\right) \tag{10}$$

where

$$T_g = T_{g0} - \frac{\dot{w}_p}{\dot{w}_g}\frac{1}{C_{pg}}\left[(h_p - h_{p0}) + \frac{1}{2}u_p^2\right] - \frac{1}{2}\frac{u_g^2}{C_{pg}} \tag{11}$$

The above equations are valid for all one dimensional gas-particle flows within the limitations of the above assumptions.

It is noted that the second term of equation (8) is always positive for accelerating flows. Thus at the nozzle throat the gas Mach number must always be less than one ($M^* < 1$) for all gas-particle flows. It is seen that the throat conditions will depend on the particle lags at the throat and hence on the nozzle inlet geometry. The nozzle inlet geometry will be of much greater importance in the design of a nozzle for use with a gas-particle system than with a pure gas system. It is possible that one may be able to alter the performance of a gas-particle nozzle by changing only the inlet geometry.

Equations (9) and (10) govern the rate of change of particle velocity and thermal energy. Because of this, the general solution of the above equations will involve a characteristic distance. Thus not only the area ratio, but the rate of change of the area ratio will determine the flow of a gas particle system. On a one dimensional basis, two gas-particle nozzles will be similar only if they have the same area ratio variation with length and hence the same length for a given expansion ratio.

CONSTANT FRACTIONAL LAG NOZZLES

Let us now limit ourselves to flows in which the particles are always in Stokes flow regime and do not undergo a phase change. For this case,

$$C_D = \frac{24}{Re} = \frac{12\mu_g}{\rho_g r_p |u_g - u_p|} = \frac{12\mu_g u_g A}{\dot{w}_g r_p |u_g - u_p|} \qquad (12)$$

$$Nu = 2 \quad , \quad \ell = \frac{k_g}{r_p} \qquad (13)$$

$$h_{po} - h_p = C_{pL}(T_{go} - T_p) \qquad (14)$$

where it has been assumed that the gas and particles have the same temperature when the system is at rest.

It has been found that there exists a family of exact solutions to the above equations for gas-particle flows in which

$$\frac{u_p}{u_g} = K \quad , \quad 0 \leq K \leq 1 \quad , \qquad (15)$$

$$\frac{T_{go} - T_p}{T_{go} - T_g} = L \quad , \quad 0 \leq L \leq 1 \quad , \qquad (16)$$

where K and L are constants. It will be shown later that these constants are determined by the axial gas velocity gradient.

Substituting the above equations into equations (8), (9),(10) and (11), we find that the equations governing the flow of a gas-particle system through a constant fractional lag nozzle are:

$$\frac{u_g}{\gamma R T_g}\left[1 + \frac{\dot{w}_p}{\dot{w}_g}\left\{K\left[\gamma(1-K) + K\right] + (\gamma - 1)\frac{C_{pL}}{C_{pg}} L \left[\frac{1 + \frac{\dot{w}_p}{\dot{w}_g}K^2}{1 + \frac{\dot{w}_p C_{pL} L}{\dot{w}_g C_{pg}}}\right]\right\} - \frac{\gamma R T_g}{u_g^2}\right]\frac{du_g}{dx} = \frac{1}{A}\frac{dA}{dx} \qquad (17)$$

$$\frac{du_g}{dx} = \frac{9}{2}\,\frac{\mu_g(1-K)}{m_p r_p^2 K^2} \tag{18}$$

$$\frac{du_g}{dx} = \frac{3}{2}\,\frac{k_g(1-L)}{m_p r_p^2 C_{pl} L K} \tag{19}$$

where

$$T_g = T_{g0} - \frac{1}{2}\,\frac{u_g^2}{C_{pg}}\left[\frac{1+\frac{\omega_p}{\omega_g}K^2}{1+\frac{\omega_p C_{pl}}{\omega_g C_{pg}}L}\right] \tag{20}$$

As equations (18) and (19) must be identical, we note that the relationship between the particle thermal lag (L) and the particle velocity lag (K) is given by

$$\frac{1-L}{L} = \frac{3\mu_g C_{pl}}{k_g}\,\frac{(1-K)}{K} = 3\,P_r\,\frac{C_{pl}}{C_{pg}}\,\frac{(1-K)}{K} \tag{21}$$

where Pr is the gas Prandtl number. In most cases of engineering interest $Pr\,\frac{C_{p_l}}{C_{p_g}} > \frac{1}{3}$ so that $L < K$ and the particle thermal lag is greater than the velocity lag. Through use of the following substitutions,

$$B = \frac{1+\frac{\omega_p}{\omega_g}K^2}{1+\frac{\omega_p}{\omega_g}\frac{C_{pl}}{C_{pg}}L} \tag{22}$$

$$C = 1 + \frac{\omega_p}{\omega_g}\left\{K\left[\gamma(1-K)+K\right] + (\gamma-1)\frac{C_{pl}}{C_{pg}}L\left[\frac{1+\frac{\omega_p}{\omega_g}K^2}{1+\frac{\omega_p}{\omega_g}\frac{C_{pl}}{C_{pg}}L}\right]\right\} \tag{23}$$

$$\bar{M} = \left[\frac{C}{\gamma R T_g}\right]^{1/2} u_g = C^{1/2} M \tag{24}$$

$$\bar{\gamma} = 1 + (\gamma-1)\frac{B}{C} \tag{25}$$

the equations governing the flow of a gas-particle system through a constant fractional lag nozzle become

$$\frac{dA}{A} = (\bar{M}^2 - 1)\frac{du_g}{u_g} \tag{26}$$

$$\frac{du_g}{dx} = \frac{9}{2}\frac{\mu_g(1-K)}{m_p r_p^2 K^2} \tag{18}$$

Solving these equations, we find that

$$\frac{A^2}{A^{*2}} = \frac{1}{\bar{M}^2}\left[\frac{2}{\bar{\gamma}+1}\left(1 + \frac{\bar{\gamma}-1}{2}\bar{M}^2\right)\right]^{\frac{\bar{\gamma}+1}{\bar{\gamma}-1}} \tag{27}$$

$$\frac{T_{g_0}}{T_g} = 1 + \frac{\bar{\gamma}-1}{2}\bar{M}^2 \tag{28}$$

$$\frac{\rho_{g_0}}{\rho_g} = \left[1 + \frac{\bar{\gamma}-1}{2}\bar{M}^2\right]^{\frac{1}{\bar{\gamma}-1}} \tag{29}$$

$$\frac{P_{g_0}}{P_g} = \left[1 + \frac{\bar{\gamma}-1}{2}\bar{M}^2\right]^{\frac{\bar{\gamma}}{\bar{\gamma}-1}} \tag{30}$$

$$\frac{u_g}{(u_g)_{max}} = \left[\frac{(\bar{\gamma}-1)\bar{M}^2}{2+(\bar{\gamma}-1)\bar{M}^2}\right]^{1/2} \tag{31}$$

$$u_g = \frac{9}{2}\frac{\mu_g(1-K)}{m_p r_p^2 K^2}\, x \tag{32}$$

where

$$(u_g)_{max} = \left\{2C_{p_g}T_{g_0}\left[\frac{1 + \frac{\dot{\omega}_p C_{p\ell} L}{\dot{\omega}_g C_{p_g}}}{1 + \frac{\dot{\omega}_p}{\dot{\omega}_g}K^2}\right]\right\}^{1/2} \tag{33}$$

The particle velocity and temperature are given by equations (15) and (16).

It is noted that equations (27) through (31) are the one dimensional gas flow equations *$(\overline{1})$ except that $\bar{\gamma}$ and $\overline{M}$ replace γ and M. With these modifications, one can use gas tables to predict the flow of a gas particle system through a constant fractional lag nozzle if the particles do not undergo a phase change.

Equation (26) shows that $\overline{M} = 1$ at the throat of a constant fractional lag nozzle. From equation (24) we find that at the throat,

$$M^{*} = C^{-\frac{1}{2}} = \left[1 + \frac{\dot{w}_p}{\dot{w}_g}\left\{K\left[\gamma(1-K)+K\right] + (\gamma-1)\frac{C_{pl}}{C_{pg}}L\left[\frac{1+\frac{\dot{w}_p}{\dot{w}_g}K^2}{1+\frac{\dot{w}_p C_{pl} L}{\dot{w}_g C_{pg}}}\right]\right\}\right]^{-\frac{1}{2}} \tag{34}$$

We see that $M^{*} < 1$ unless there is no particle flow ($K = L = 0$ or $\frac{\dot{w}_p}{\dot{w}_g} = 0$).

We note that the throat conditions depend on the particle lags and hence on the nozzle inlet geometry as was shown earlier.

Through use of equations (27), (31) and (32), one can show that the area variation in a constant fractional lag nozzle is given by

$$\frac{A}{A^{*}} = \left[\frac{\bar{\gamma}-1}{\bar{\gamma}+1}\right]^{1/2}\left[\frac{2}{\bar{\gamma}+1}\right]^{\frac{1}{\bar{\gamma}-1}}\frac{1}{z}\left[\frac{1}{1-z^2}\right]^{\frac{1}{\bar{\gamma}-1}} \tag{35}$$

where

$$z = \frac{x}{x_{max}} = \frac{u_g}{(u_g)_{max}} = \frac{9\mu_g(1-K)}{2m_p r_p^2 K^2}\frac{x}{(u_g)_{max}} \tag{36}$$

* Barred numbers in parenthesis refer to the references at the end of this report.

Figure I is a plot of $\frac{y}{y_t}$ vs z for a two dimensional constant fractional lag nozzle and Figure II is a plot of $\frac{r}{\bar{r}_t}$ vs z for an axially symmetrical constant fractional lag nozzle. Both of the above figures represent nozzles for flows in which $\bar{\gamma} = 1.20$.

In a two dimensional nozzle,

$$\frac{A}{A^*} = \frac{y}{y_T} \tag{37}$$

and the radius of curvature of the nozzle wall at the throat is given by

$$R_T = \frac{1}{\left[1+\left(\frac{dy}{dx}\right)^2_{y_g=y_g^*}\right]^{3/2}\left(\frac{d^2y}{dx^2}\right)_{y_g=y_g^*}} \tag{38}$$

One can show from equations (35), (36), (37) and (38) that in a two dimensional constant fractional lag nozzle,

$$\frac{R_T}{y_T} = \frac{u_g^{*2}}{(\bar{\gamma}+1)y_T^2\left[\frac{du_g}{dx}\right]^2} = \frac{1}{\bar{\gamma}+1}\left[\frac{2}{9}\,\frac{m_p r_p^2 K^2}{\mu_g(1-K)}\,\frac{u_g^*}{y_T}\right]^2 \tag{39}$$

Similarly, one can show that in an axially symmetrical constant fractional lag nozzle,

$$\frac{R_T}{r_T} = \frac{2u_g^{*2}}{(\bar{\gamma}+1)r_T^2\left[\frac{du_g}{dx}\right]^2} = \frac{2}{\bar{\gamma}+1}\left[\frac{2}{9}\,\frac{m_p r_p^2 K^2}{\mu_g(1-K)}\,\frac{u_g^*}{r_T}\right]^2 \tag{40}$$

It has been shown ($\bar{2}$) that Sauer's ($\bar{3}$) relationship between nozzle geometry and the axial velocity gradient (for a pure gas flow) can be derived in a similar way from one dimensional flow consideration. In particular, the middle terms in equations (39) and (40) are the same as those obtained by Sauer (with $\bar{\gamma}$ replacing γ). Equations (39) and (40) are thus an extension of Sauer's results to the case of gas-particle flows in constant fractional lag nozzles.

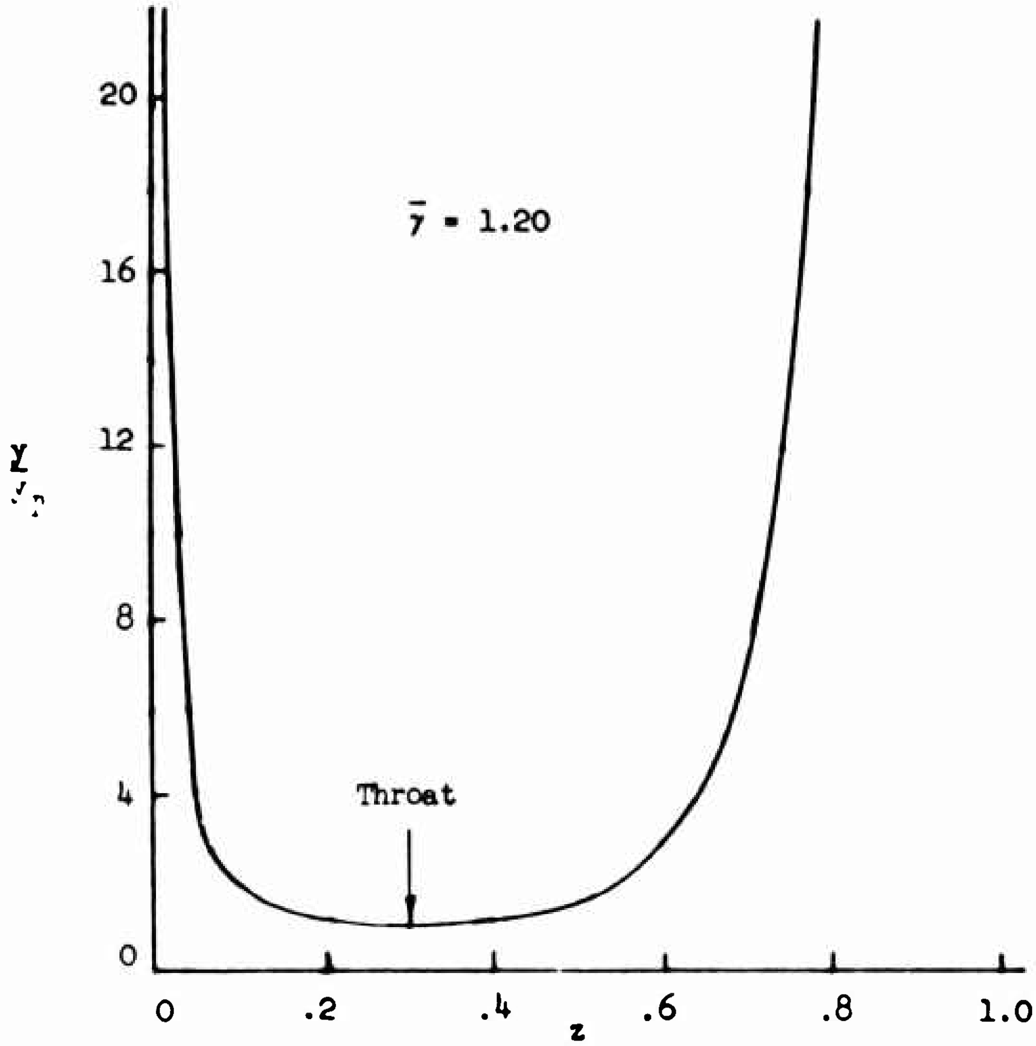

Figure I - Two Dimensional Constant Fractional Lag Nozzle

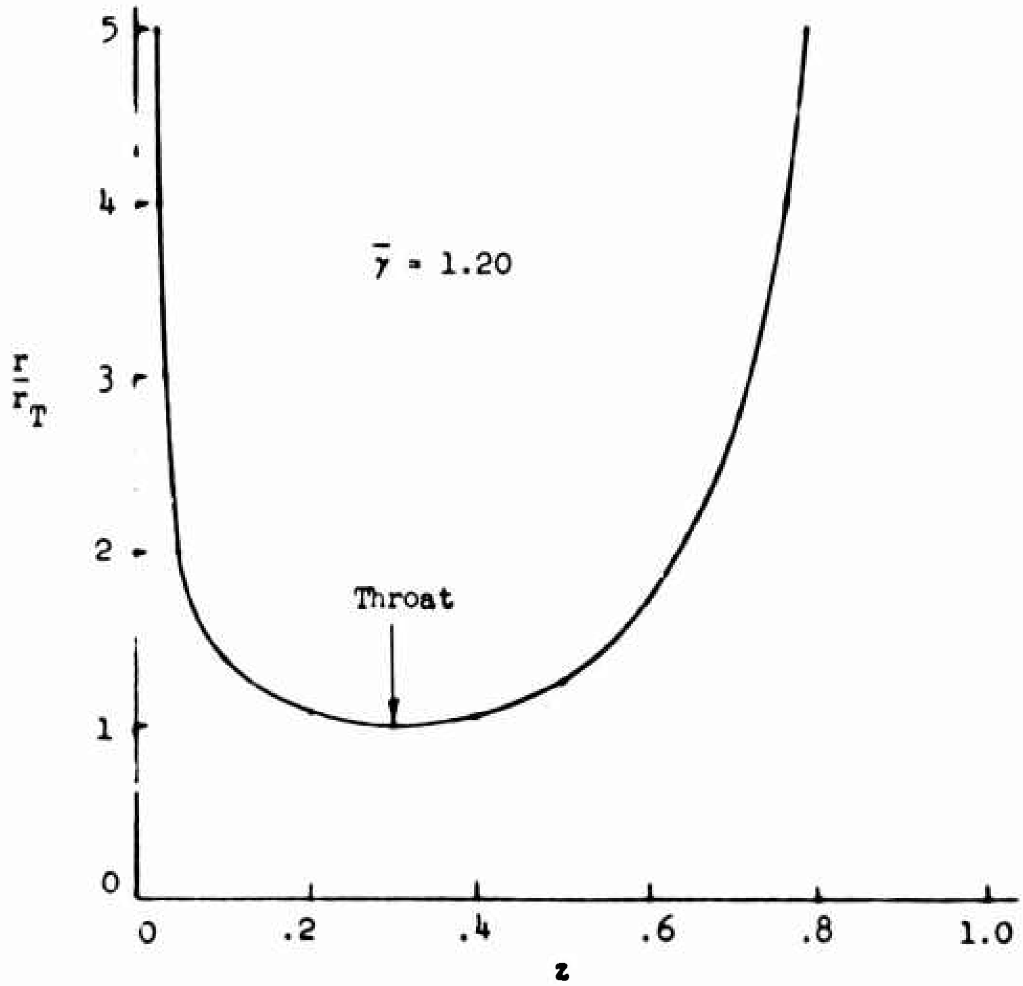

Figure II - Axially Symmetric Constant Fractional Lag Nozzle

Equation (18) shows that

$$\frac{du_g}{dx} = \frac{9}{2}\frac{\mu_g(1-K)}{m_p r_p^2 K^2} = \text{constant} \tag{18}$$

in a constant fractional lag nozzle. As this condition is approximately true for a distance through the throat of most nozzles of engineering interest, it would appear that the above results have general applicability in the throat region of most nozzles.

Equations (18) and (19) show that the particle lags are determined by the axial gas velocity gradient in a constant fractional lag nozzle. One sees that to decrease the particle lags one must decrease the axial gas velocity gradient. As the axial gas velocity gradient is greatest near the throat of a nozzle, it would appear that the throat region is the critical region in the design of a nozzle for a gas particle system.

Equations (39) and (40) give the radius of curvature of the nozzle wall at the throat in terms of the throat conditions, particle sizes and particle velocity lag for two dimensional and axially symmetrical constant fractional lag nozzles. It is seen that for fixed particle lags, the radius of curvature of the nozzle wall at the throat is proportional to the fourth power of the particle size.

Figures I and II show that constant fractional lag nozzles are approximately symmetrical through the throat. Most conventional gas nozzles are asymmetrical through the throat, having a smaller wall radius of curvature downstream of the throat, than upstream. The axial gas velocity gradient in these nozzles is higher downstream of the throat than upstream. It would appear from the above analysis that this type nozzle would cause a performance loss when used with a gas particle system. The smaller downstream radius of curvature would partially nullify the benefits gained by having the larger upstream radius of curvature. Because of this it is felt that nozzles designed for a gas-particle system should have a constant radius of curvature through the throat.

Equation (32) shows that there is a characteristic distance associated with the one dimensional flow of a gas-particle system through a constant fractional lag nozzle. One sees from this equation that the nozzle lengths required from chamber to throat and from throat to exit are

$$L_{cT} = \frac{2}{9}\,\frac{m_p r_p^2 K^2}{\mu_g(1-K)}\,u_g^* \tag{41}$$

and

$$L_{Te} = \frac{2}{9}\,\frac{m_p r_p^2 K^2}{\mu_g(1-K)}\left(u_{ge} - u_g^*\right) \tag{42}$$

in a constant fractional lag nozzle. These lengths are probably the minimum lengths one should consider in designing a nozzle for a gas-particle system. Both the lengths available from chamber to throat and throat to exit must be considered when designing a nozzle for a gas-particle exhaust system. This is a consequence of the fact that the particle velocity at any point depends on the particle's acceleration history from the chamber. It is seen that for fixed particle lags, the nozzle length is proportional to the particle size squared.

One is generally interested in the most compact nozzle (short and light weight) which will give the desired performance. Because of the strong dependence of the nozzle length (and hence weight) on the particle size, there is great incentive to insure that the particle size is small. As the combustion chamber geometry will determine the particle size entering the nozzle, the design of an optimum nozzle for use with a gas-particle exhaust system must begin in the combustion chamber. One cannot assume that the effect of the combustion chamber and nozzle inlet geometry will have little effect on the performance of a rocket in which the exhaust is a gas-particle system.

PARTICLE SOLIDIFICATION

If the particle system undergoes a phase change during expansion through the nozzle (solidification), one can extend the above analysis (for a constant fractional lag nozzle) to include this case. Let us assume that the particle temperature remains constant during solidification. From equations (4), (7) and (32), we find that the equation governing the gas temperature during particle solidification is

$$u_g \frac{dT_g}{dx} - \frac{3 k_g \dot{w}_p}{\dot{w}_g C_{pg} m_p r_p^2 K}\left(T_{pm} - T_g\right) + \frac{9}{2}\frac{\mu_g(1-K)}{m_p r_p^2 K^2}\left(1 + \frac{\dot{w}_p}{\dot{w}_g}K^2\right)\frac{u_g^2}{C_{p_g}} = 0 \qquad (43)$$

Solving this equation with the boundary conditions that the gas temperature at the beginning of particle solidification ($T_p = T_{pm}$, $U_g = U_{gm}$, $T_g = T_{gm}$) is that given by equation (11), we find that during particle solidification,

$$T_g = T_{pm} - \left[\frac{1 + \frac{\dot{w}_p}{\dot{w}_g}K^2}{1 + \frac{1}{3}\frac{\dot{w}_p}{\dot{w}_g}\frac{K}{Pr(1-K)}}\right]\frac{u_g^2}{2C_{pg}} - \left\{T_{pm} - T_{gm} - \left[\frac{1 + \frac{\dot{w}_p}{\dot{w}_g}K^2}{1 + \frac{1}{3}\frac{\dot{w}_g}{\dot{w}_p}\frac{K}{Pr(1-K)}}\right]\frac{u_{gm}^2}{2C_{pg}}\right\}\left[\frac{u_g}{u_{gm}}\right]^{-\frac{2}{3}\frac{\dot{w}_p}{\dot{w}_g}\frac{K}{Pr(1-K)}} \qquad (44)$$

where

$$T_{gm} = T_{g0} - \frac{\left(T_{g0} - T_{pm}\right)}{L} \qquad (45)$$

$$u_{gm} = \left\{\left[\frac{1 + \frac{\dot{w}_p}{\dot{w}_g}\frac{C_{pl}L}{C_{pg}}}{1 + \frac{\dot{w}_p}{\dot{w}_g}K^2}\right]\frac{2C_{pg}\left(T_{g0} - T_{pm}\right)}{L}\right\}^{1/2} \qquad (46)$$

We note that the gas undergoes an isothermal expansion ($T_g = T_{p_m}$) only if there is no particle lag ($K = 1$). From equations (16), (27), (28) and (29), we find that the area ratio at which particle solidification first occurs is given by

$$\frac{A}{A^*} = \frac{u_g^*}{u_{gm}} \left[\frac{T_g^* L}{T_{pm} - (1-L) T_{go}} \right]^{\frac{1}{\gamma-1}} \qquad (47)$$

In most cases of engineering interest, $T_{pm} < T_g^*$ so that particle solidification can occur only in the divergent section of the nozzle. In this case, one can show that the area ratio at which particle solidification begins is a minimum for no particle lag (L = 1), increases as the particle lag increases ($L \rightarrow 0$) and becomes infinite when

$$L = 1 - \frac{T_{pm}}{T_{go}} \qquad (48)$$

This is the minimum value of L for which solidification will occur in any nozzle. As the addition of energy in the divergent section of a nozzle is more effective in increasing performance the lower the expansion ratio at which the energy is added, it is clear that one must have a low lag nozzle if one is to make most efficient use of the particle solidification energy. It is also clear that if the particle lag is large enough, one will be unable to utilize the particles heat of fusion as the particles will solidify outside the nozzle.

It is noted that in principle one can use equation (44) for the gas temperature and obtain expressions for the other quantities of interest during particle solidification. It appears however that the required integrals cannot be expressed in closed form for the general case of arbitrary particle lag due to the last term on the right hand side of equation (44). Because of this, it is felt that any further investigation of this case must be done numerically.

It is found however that for the case of no particle lag, one can extend the above investigation. In the interest of completeness, this has been done although in principle this case can only be obtained in an infinitely long nozzle. During particle solidification ($T_g = T_p = T_{pm}$, $h_{ps} \leq h_p \leq h_{pl}$), equation (8) becomes

$$\frac{u_g}{RT_{pm}}\left[1+\frac{\dot{\omega}_p}{\dot{\omega}_g}-\frac{RT_{pm}}{u_g^2}\right]\frac{du_g}{dx} = \frac{1}{A}\frac{dA}{dx} \tag{49}$$

Solving this equation we find that during particle solidification,

$$\frac{A}{A_m} = \frac{u_{gm}}{u_g}\, e^{\frac{\dot{\omega}_p}{\dot{\omega}_g}\frac{(h_{pl}-h_p)}{RT_{pm}}} \tag{50}$$

$$T_g = T_{pm} \tag{51}$$

$$\frac{\rho_g}{\rho_{gm}} = e^{-\frac{\dot{\omega}_p}{\dot{\omega}_g}\frac{(h_{pl}-h_p)}{RT_{pm}}} \tag{52}$$

$$\frac{P_g}{P_{gm}} = e^{-\frac{\dot{\omega}_p}{\dot{\omega}_g}\frac{(h_{pl}-h_p)}{RT_{pm}}} \tag{53}$$

$$u_g = \left[u_{gm}^2 + 2\frac{\dot{\omega}_p}{\dot{\omega}_g}\frac{(h_{pl}-h_p)}{\left(1+\frac{\dot{\omega}_p}{\dot{\omega}_g}\right)}\right]^{1/2} \tag{54}$$

The conditions at the beginning of particle solidification are obtained by substituting

$$K = L = 1, \quad T_g = T_p = T_{pm} \tag{55}$$

into equations (22) through (25) and (27) through (31). At the end of particle solidification,

$$h_{pl} - h_p = h_{pl} - h_{ps} = \Delta H_f . \tag{56}$$

The conditions existent at the end of particle solidification can be found by substituting this relationship into the above equations.

After particle solidification ($h_p \leq h_{ps}$), the equation governing the flow in the no lag case is

$$\frac{u_g}{\gamma R T_g}\left[1+\frac{\dot{w}_p}{\dot{w}_g}\left\{1+(\gamma-1)\frac{C_{ps}}{C_{pg}}\left[\frac{1+\frac{\dot{w}_p}{\dot{w}_g}}{1+\frac{\dot{w}_p C_{ps}}{\dot{w}_g C_{pg}}}\right]\right\}-\frac{\gamma R T_g}{u_g^2}\right]\frac{du_g}{dx} = \frac{1}{A}\frac{dA}{dx} \tag{57}$$

where

$$T_g = T_{pm} - \frac{1}{2C_{pg}}\left[\frac{1+\frac{\dot{w}_p}{\dot{w}_g}}{1+\frac{\dot{w}_p C_{ps}}{\dot{w}_g C_{pg}}}\right]\left(u_g^2-u_{gs}^2\right) \tag{58}$$

Through use of the following substitutions,

$$B_1 = \frac{1+\frac{\dot{w}_p}{\dot{w}_g}}{1+\frac{\dot{w}_p C_{ps}}{\dot{w}_g C_{pg}}} \tag{59}$$

$$C_1 = 1+\frac{\dot{w}_p}{\dot{w}_g}\left\{1+(\gamma-1)\frac{C_{ps}}{C_{pg}}\left[\frac{1+\frac{\dot{w}_p}{\dot{w}_g}}{1+\frac{\dot{w}_p C_{ps}}{\dot{w}_g C_{pg}}}\right]\right\} \tag{60}$$

$$\bar{\gamma}_s = 1+(\gamma-1)\frac{B_1}{C_1} \tag{61}$$

we find that after particle solidification,

$$\frac{A}{A_s} = \frac{u_{gs}}{u_g}\left[\frac{T_{pm}}{T_g}\right]^{\frac{1}{\bar{\gamma}_s-1}} \tag{62}$$

$$\frac{\rho_{gs}}{\rho_g} = \left[\frac{T_{pm}}{T_g}\right]^{\frac{1}{\bar{\gamma}_s-1}} \tag{63}$$

$$\frac{P_{gs}}{P_g} = \left[\frac{T_{pm}}{T_g}\right]^{\frac{\bar{\gamma}_s}{\bar{\gamma}_s-1}} \tag{64}$$

$$u_g = \left\{u_{gs}^2 + 2C_{pg}\left[\frac{1+\frac{\dot{w}_p C_{ps}}{\dot{w}_g C_{pg}}}{1+\frac{\dot{w}_p}{\dot{w}_g}}\right]\left(T_{pm}-T_g\right)\right\}^{1/2} \tag{65}$$

It should be remembered that the above relationships (equations (50) through (65)) hold only for the case of no particle lag (K = L = 1).

NOZZLE PERFORMANCE

One measure of nozzle performance is the specific impulse which for a fully expanded nozzle is defined as

$$I_{sp} = \frac{1}{g\left(1 + \frac{\dot{w}_p}{\dot{w}_g}\right)} \left[u_g + \frac{\dot{w}_p}{\dot{w}_g} u_p \right] \tag{66}$$

for a gas-particle system expanding through a nozzle. From the above analysis we find that for a constant fractional lag nozzle in which the particles do not undergo a phase change,

$$I_{sp} = \frac{\left(1 + \frac{\dot{w}_p}{\dot{w}_g} K\right)}{g\left(1 + \frac{\dot{w}_p}{\dot{w}_g}\right)} \sqrt{\left[\frac{1 + \frac{\dot{w}_p}{\dot{w}_g} \frac{C_{p\ell} L}{C_{pg}}}{1 + \frac{\dot{w}_p}{\dot{w}_g} K^2}\right] 2 C_{pg} \left(T_{g0} - T_{ge}\right)} \tag{67}$$

Altman and Carter (4) give expressions for the specific impulse of a gas particle system for the following special cases:

1) Complete kinetic and thermal equilibrium (K = L = 1)
2) Kinetic equilibrium, thermal insulation (K = 1, L = 0)
3) Thermal equilibrium, kinetic non-equilibrium (K = 0, L = 1)
4) Complete insulation (K = L = 0)

The above expression for the specific impulse reduces to Altman and Carter's expressions for these cases. It should perhaps be pointed out that these limiting cases correspond to the following physical conditions:

1) The complete kinetic and thermal equilibrium case can occur only in an infinitely long nozzle.

2) The kinetic equilibrium, thermal insulation case can occur only in an infinitely long nozzle and only if the thermal conductivity of the particles is zero.

3) The thermal equilibrium, kinetic non-equilibrium case occurs only in an infinitely long nozzle and only if the particles are infinitely large.

4) The kinetic and thermal insulation case can occur only in an infinitesimally short nozzle.

Similar expressions can be given for the specific impulse for the case of no particle lag (an infinitely long nozzle) during and after particle solidification. During particle solidification ($T_g = T_p = T_{pm}$, $h_{ps} \leq h_p \leq h_{pl}$),

$$I_{sp} = \frac{1}{g}\sqrt{2\left[c_{pg}(T_{go}-T_{pm}) + \frac{\dot{w}_p}{\dot{w}_g}(h_{po}-h_{pl})\right]} \tag{68}$$

and after particle solidification ($h_p \leq h_{ps}$),

$$I_{sp} = \frac{1}{g}\sqrt{2\left[c_{pg}(T_{go}-T_{ge}) + \frac{\dot{w}_p}{\dot{w}_g}(h_{po}-c_{ps}T_{ge})\right]} \tag{69}$$

It is noted that through use of equations (27), (28)and (67) one can obtain the specific impulse as a function of expansion ratio for any particle lag in a constant fraction lag nozzle in which the particles do not undergo a phase change. By use of equations (50), (57) and (68) and equation (62), (65) and (69) one can obtain the specific impulse as a function of expansion ratio during and after particle solidification for the case of no particle lag.

One can show from the above expressions that for a fixed nozzle expansion ratio, the nozzle efficiency decreases as the particle lags increase ($I_{sp} \rightarrow 0$ as $K \rightarrow 0$ and $L \rightarrow 0$). Numerical results for typical metallized propellants show that the nozzle loss in I_{sp} is approximately given by

$$\Delta I_{sp} = (I_{sp})_{K=1} - I_{sp} \simeq \frac{\dot{w}_p}{\dot{w}_g} \frac{(1-K)(I_{sp})_{K=1}}{1+\frac{\dot{w}_p}{\dot{w}_g}} \tag{70}$$

for nozzles with large expansion ratios.

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NOMENCLATURE

A	-	Flow area
C_D	-	Particle drag coefficient
C_{pg}	-	Gas heat capacity
C_{pl}	-	Particle heat capacity ($T_p > T_{pm}$)
C_{ps}	-	Particle heat capacity ($T_p < T_{pm}$)
g	-	Gravitational constant
h	-	Film heat transfer coefficient
h_p	-	Particle heat content
h_{pl}	-	Particle heat content after melting ($T_p = T_{pm}$)
h_{ps}	-	Particle heat content before melting ($T_p = T_{pm}$)
I_{sp}	-	Specific impulse
K	-	Constant defining particle kinetic lag
k_g	-	Gas thermal conductivity
L	-	Constant defining particle thermal lag
L_{ct}	-	Nozzle length, chamber to throat
L_{te}	-	Nozzle length, throat to exit
$\bar{M}$	-	Modified Mach number for a gas-particle system
M	-	Gas Mach number
m_p	-	Particle density
Nu	-	Nusselt number, $2hr_p/k_g$
P_g	-	Gas pressure
Pr	-	Gas Prandtl number $\mu_g C_{pg}/k_g$
R	-	Gas constant

R_e - Particle Reynold's number, $2\ \rho_g r_p \left| U_g - U_p \right| / \mu_g$

R_T - Nozzle wall radius of curvature at nozzle throat

r - Nozzle wall coordinate, (axially symmetrical nozzle)

r_p - Particle radius

r_T - Nozzle throat radius, (axially symmetrical nozzle)

T_g - Gas temperature

T_p - Particle temperature

T_{pm} - Particle solidification temperature

U_g - Gas velocity

U_p - Particle velocity

$\dot{w}_g$ - Gas weight flow

$\dot{w}_p$ - Particle weight flow

x - Coordinate in flow direction

y - Nozzle wall coordinate (two dimensional nozzle)

y_T - Nozzle throat radius (two dimensional nozzle)

z - Normalized coordinate in flow direction

γ - Gas adiabatic expansion coefficient

$\bar{\gamma}$ - Expansion coefficient for gas-particle system before particle solidification

$\bar{\gamma}_s$ - Expansion coefficient for gas-particle system after particle solidification

ΔH_f - Particle heat of fusion

μ_g - Gas viscosity coefficient

ρ_g - Gas density

ρ_p - Particle density in the gas (based on gas volume)

Subscripts

e - refers to exit conditions

m - refers to conditions when particles begin to solidify

o - refers to chamber reference conditions

s - refers to conditions when particles have finished solidifying

Superscripts

$*$ - refers to throat conditions

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